A HIERARCHICAL BAYESIAN IMPLEMENTATION OF PURELY BAYESIAN AND BAYESIAN MIXTURE MODELS OF CONDITIONAL REASONING

Henrik Singmann

A girl had sexual intercourse.

How likely is it that the girl is pregnant?

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How likely is it that the girl is pregnant?

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A girl is NOT pregnant.

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Experimental paradigm:

- 1. Session: Reduced inferences (no conditional)
  2. Session: Full conditional inferences
- 4 different conditionals (i.e., contents)
- Participants respond to all 4 inferences per session and content.

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RESULTS

Balloon: If a balloon is pricked with a needle then it will pop.

- few disablers, many alternatives

Coke: If a person drinks a lot of coke then the person will gain weight.

- many disablers, many alternatives

Girl: If a girl has sexual intercourse then she will be pregnant.

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Predator: If a predator is hungry then it will search for prey.

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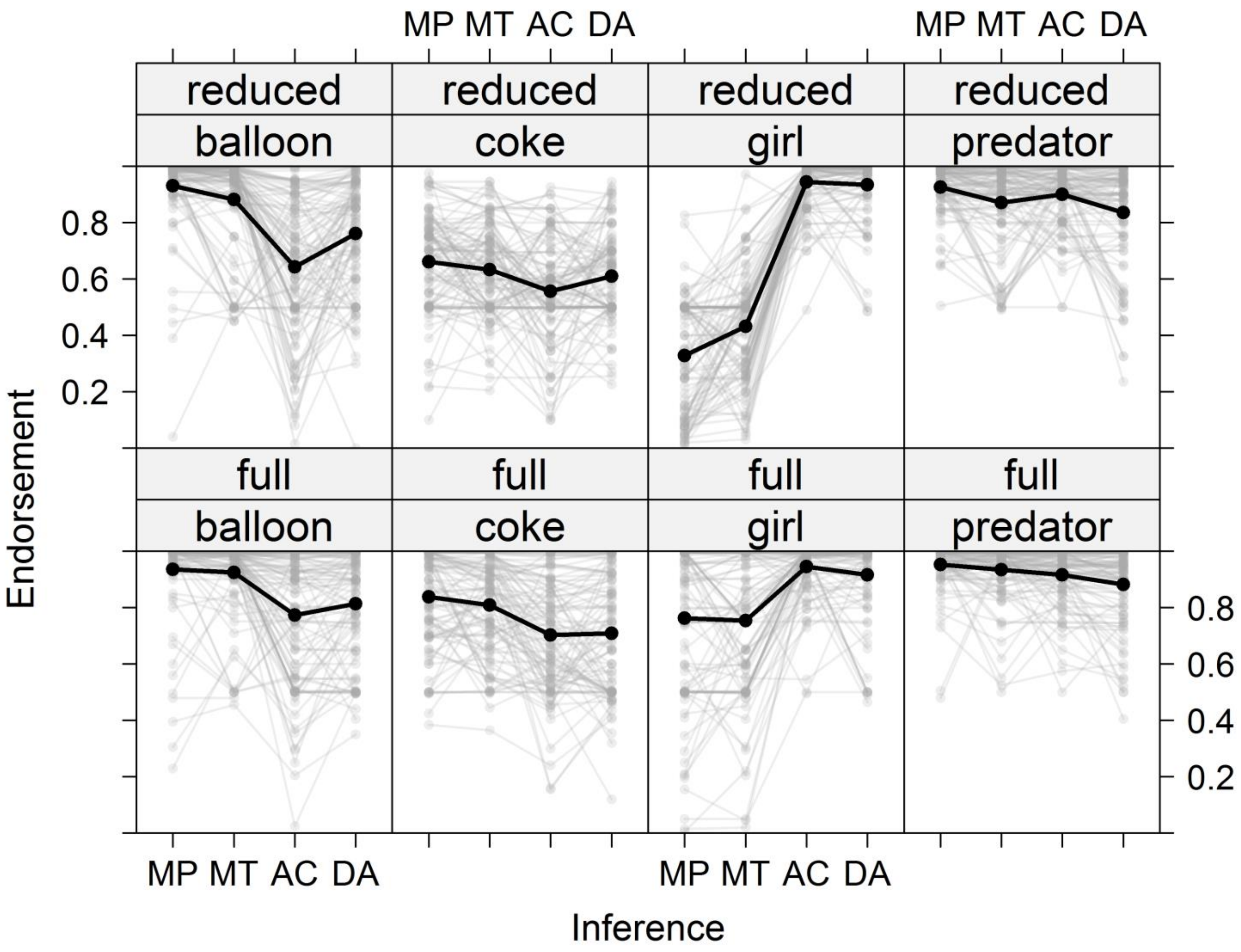
N = 101

Klauer, Beller, & Hütter (2010, Exp. 1)

Singmann, Klauer, & Beller (2016, Exp. 1 & 3)



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Joint probability distribution $F_{p,q}$		
	q	$\neg q$
p	$P(p \wedge q)$	$P(p \wedge \neg q)$
$\neg p$	$P(\neg p \wedge q)$	$P(\neg p \wedge \neg q)$

3 free parameters

Provides conditional probabilities/predictions:

- $P(MP) = P(q \mid p) = P(p \wedge q) \,/\, P(p)$
- $P(MT) = P(\neg p \mid \neg q) = P(\neg p \wedge \neg q) \,/\, P(\neg q)$
- $P(AC) = P(p \mid q) = P(p \wedge q) \,/\, P(q)$
- $P(DA) = P(\neg q \mid \neg p) = P(\neg p \wedge \neg q) \,/\, P(\neg p)$

Oaksford, Chater, & Larkin (2000)
Oaksford & Chater (2007)

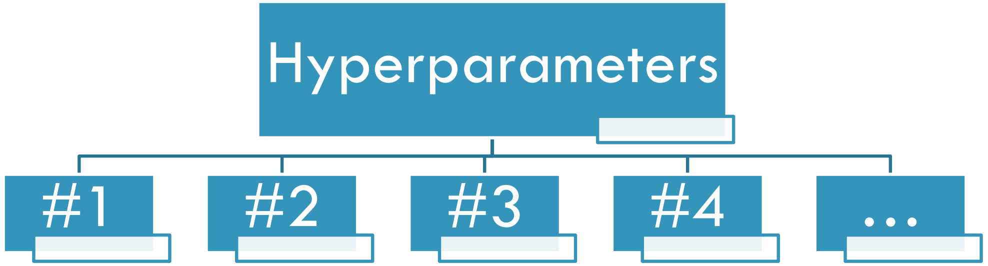
HIERARCHICAL MODELING

2 classical approaches for dealing with individual differences:

- *complete pooling*: ignores individual variability
- *no pooling*: ignores similarity across participants (e.g., Oaksford, Chater, & Larkin, 2000; Klauer, Beller, & Hütter, 2010; Singmann, Klauer, & Beller, 2016)

Partial pooling principled alternative:

- Individual level parameters are drawn from group-level distributions
- Provides higher precision for parameter estimates (even on the individual level)

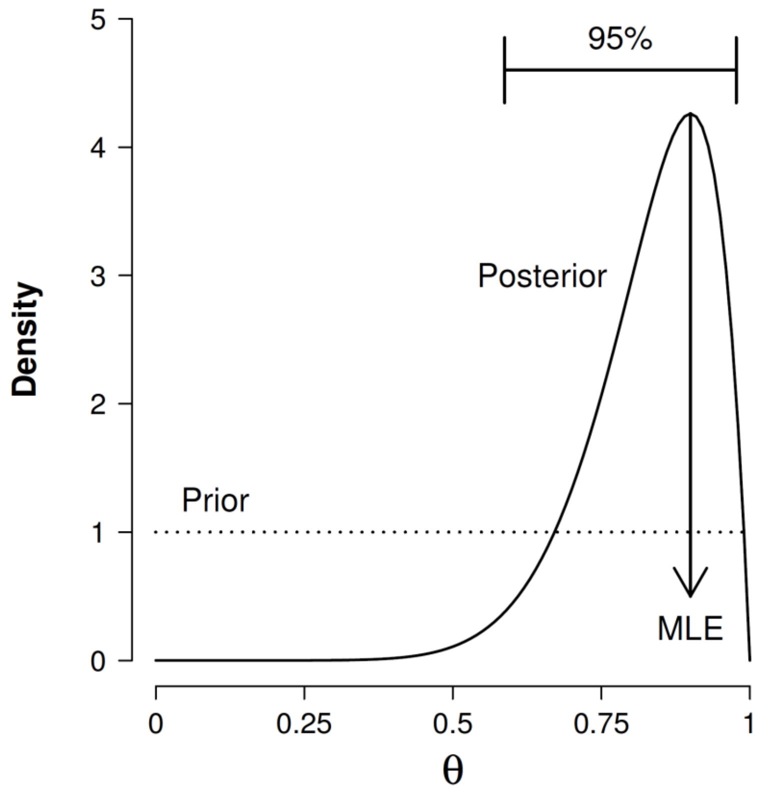


BAYESIAN STATISTICS

Requires likelihood (i.e., no least squares).

Information (uncertainty) regarding parameters expressed via (continuous) probability distributions.

1. *Prior distributions* capture ignorance before data is collected.
2. Prior distributions *updated* in light of data using *Bayes' theorem*.
3. *Posterior distributions* reflect new state of knowledge.



Ferrari & Cribari-Neto (2004)
Simas, Barreto-Souza, & Rocha (2010)

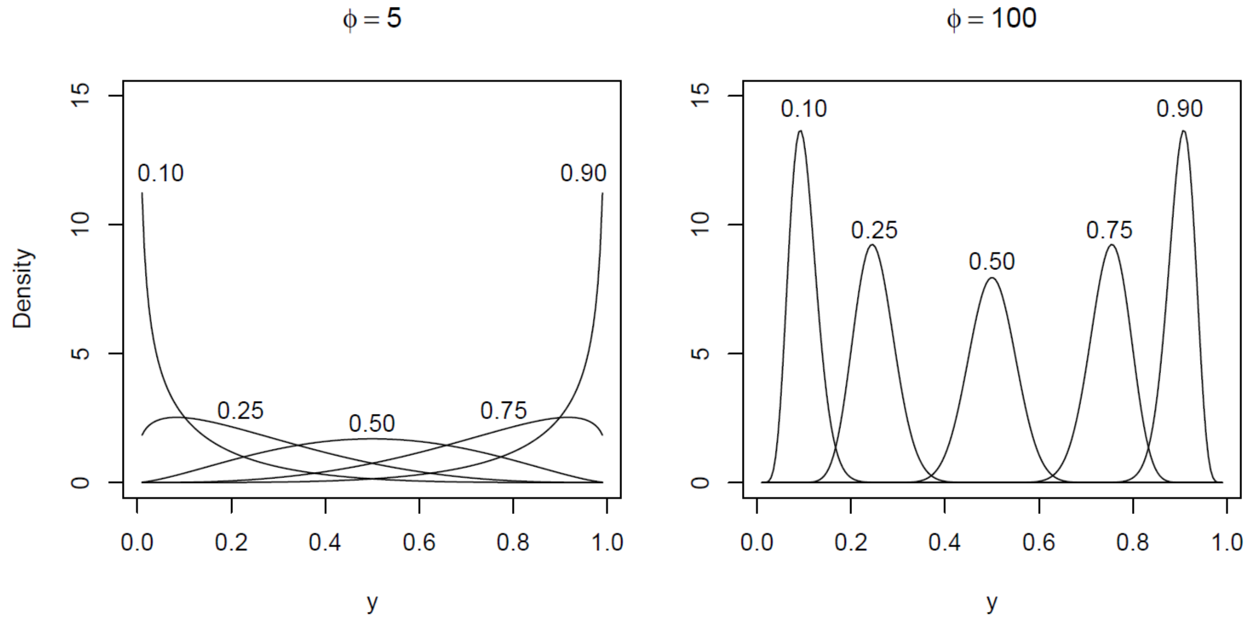
BETA REGRESSION

Allows to model data in unit interval (0, 1) using beta distribution.

Instead of shape parameters α and β, uses mean μ and precision ϕ:

- $\alpha = \mu\phi$
- $\beta = (1 - \mu)\phi$

Naturally addresses heteroscedasticity: More variation in mid ranges than at the upper and lower end.



HYPERDISTRIBUTION FOR PROBABILITY DISTRIBUTION

Predictions of Bayesian model result from probability distribution $\boldsymbol{F_{p,q}}$.

Oaksford and Chater parameterize $\boldsymbol{F_{p,q}}$ using three parameters:

- *a* = *P*(*p*)
- *b* = *P*(*q*)
- e = *P*(not-*q*|*p*) = 1- *P*(*q*|*p*)

Not all values of *a*, *b*, and e result in proper probability distribution:

- e is bound: $[\max\left(\frac{a-b}{a}, 0\right), \min\left(\frac{1-b}{a}, 1\right)]$

The joint distribution of *a*, *b*, and e cannot be a proper hyper/prior distribution for $F_{p,q}$.

Alternative provided by Dirichlet distribution, which usually has 2 parameters:

- $K \geq 2$, number of categories (integer)
- $\alpha_1, \ldots, \alpha_K$, concentration parameter

Support over K-dimensional vectors that sum to 1 (i.e., $(K-1)$-dimensional simplex).

Parameterization as in beta-regression possible (e.g., Kemp, Perfors, & Tenenbaum, 2007):

- $\boldsymbol{\gamma}$: mean of hyperparameter
- ψ: precision of hyperparameter

HIERARCHICAL BAYESIAN BAYESIAN MODEL

(simple model)

Data: $E_{\text{red},ijk} \sim \text{Beta}(\alpha_{\text{red},ijk}, \beta_{\text{red},ijk})$

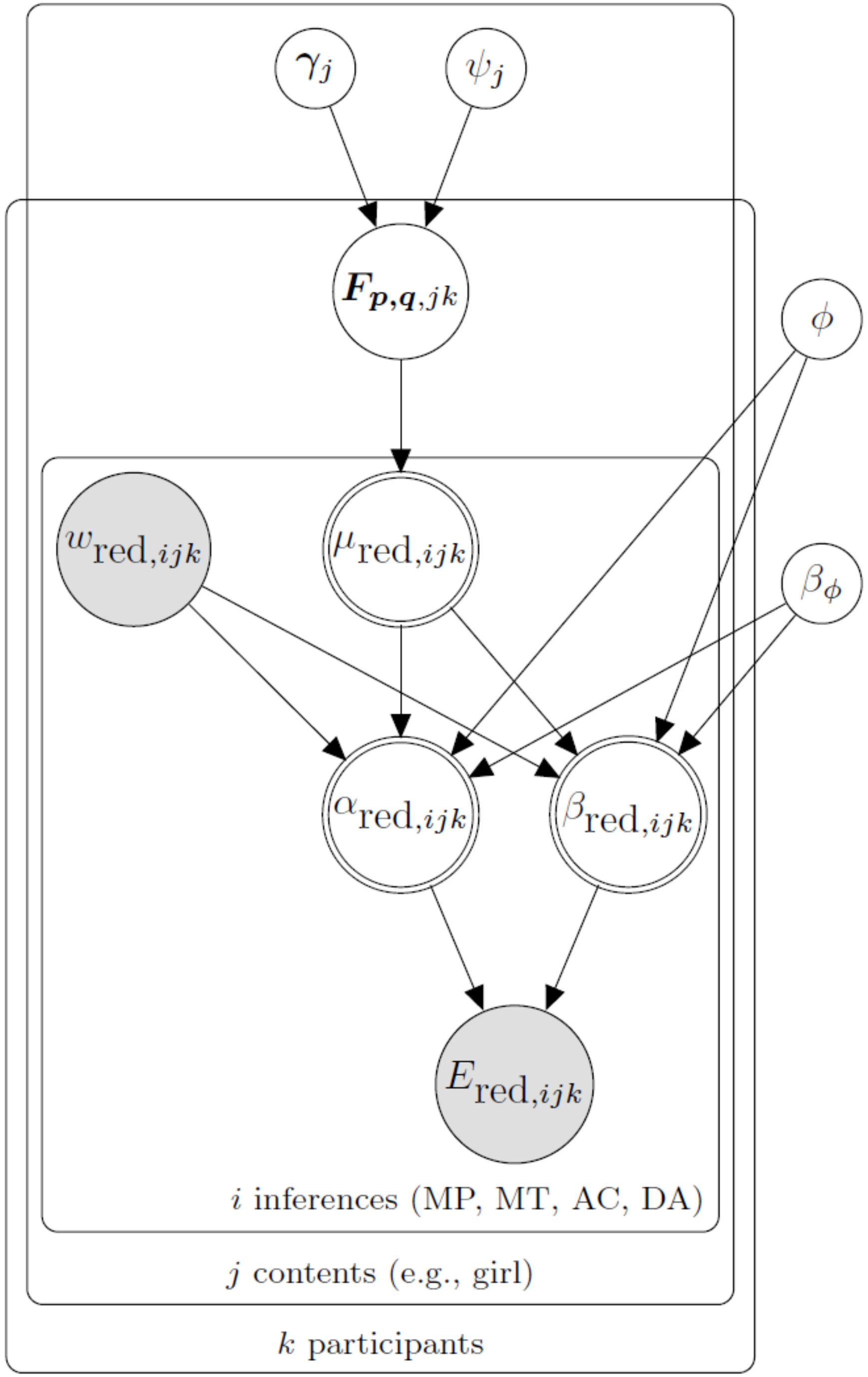
Group-level distribution: $\boldsymbol{F}_{\boldsymbol{p},\boldsymbol{q},jk} \sim \text{Dirichlet}(\gamma_j \times \psi_j)$

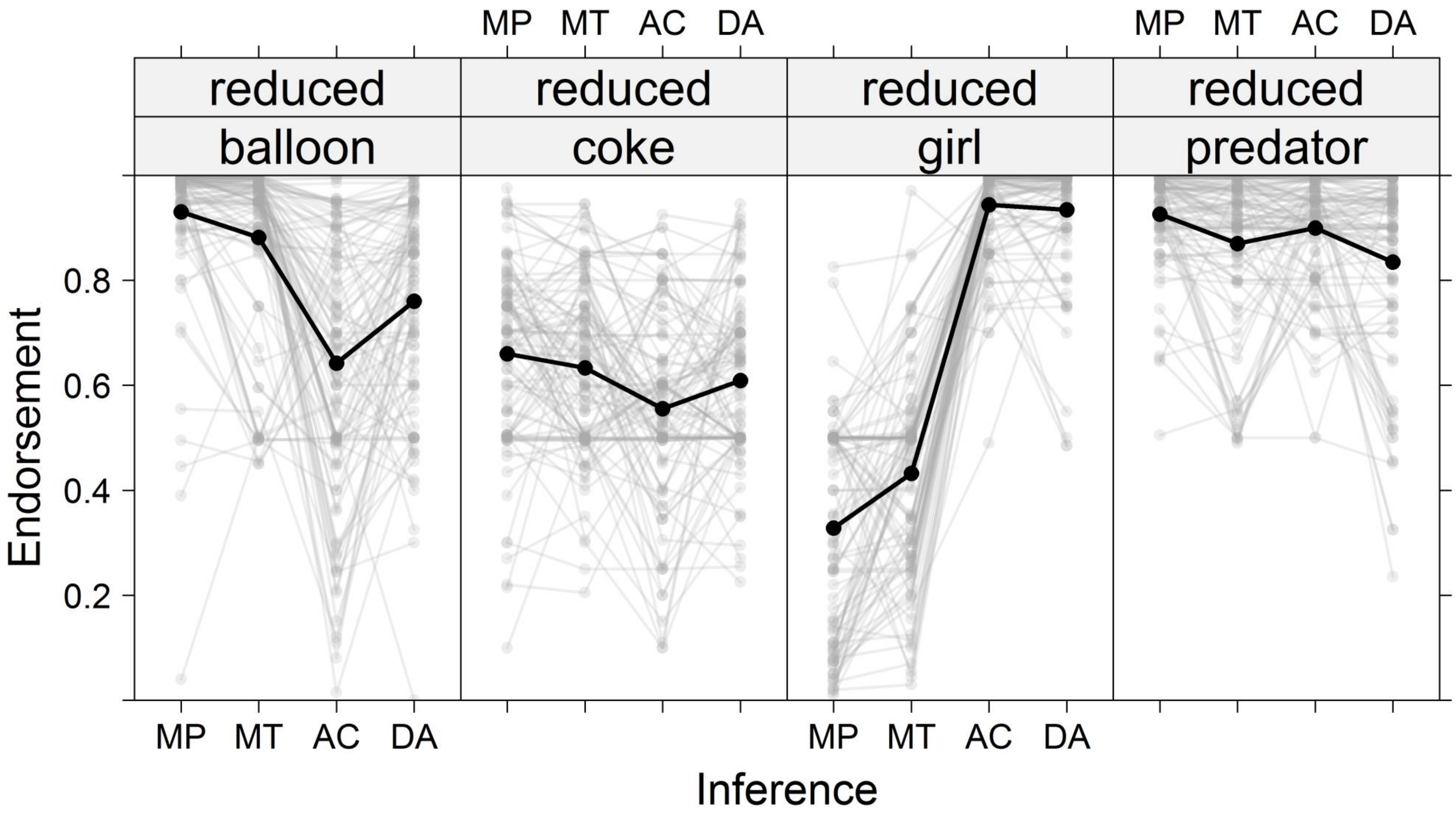
Priors:

$$\gamma_j \sim \text{Dirichlet}(1)$$
$$\psi_j \sim \text{Cauchy}^+(1, 25)$$
$$\phi \sim \text{Cauchy}^+(2, 25)$$
$$\beta_\phi \sim \text{Cauchy}^+(0, 25)$$

Beta regression:

$$\alpha = \mu \times (\phi + \beta_\phi w)$$
$$\beta = (1 - \mu) \times (\phi + \beta_\phi w)$$





reduced
balloon
reduced
coke
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girl
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predator
MP
MT
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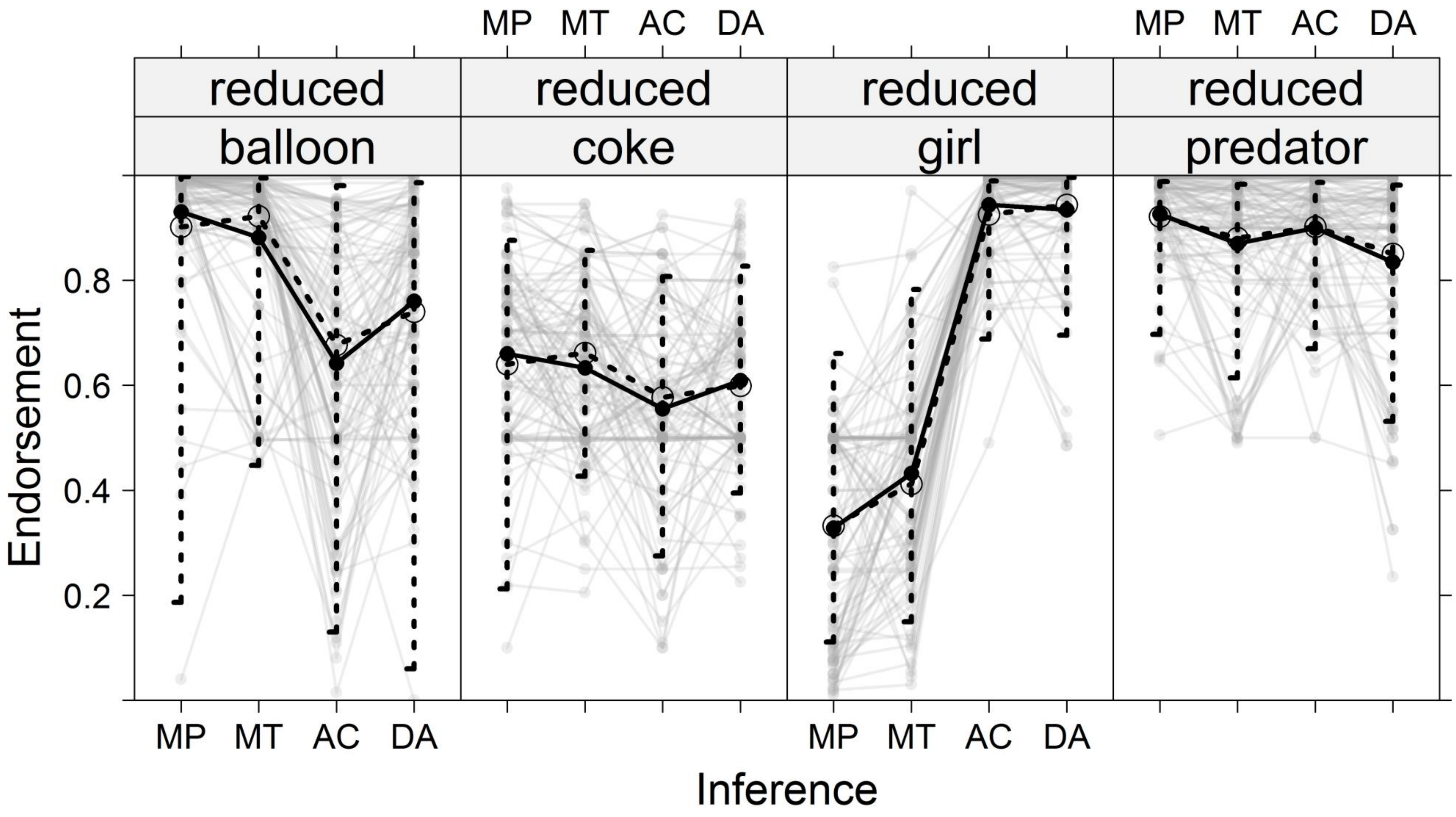
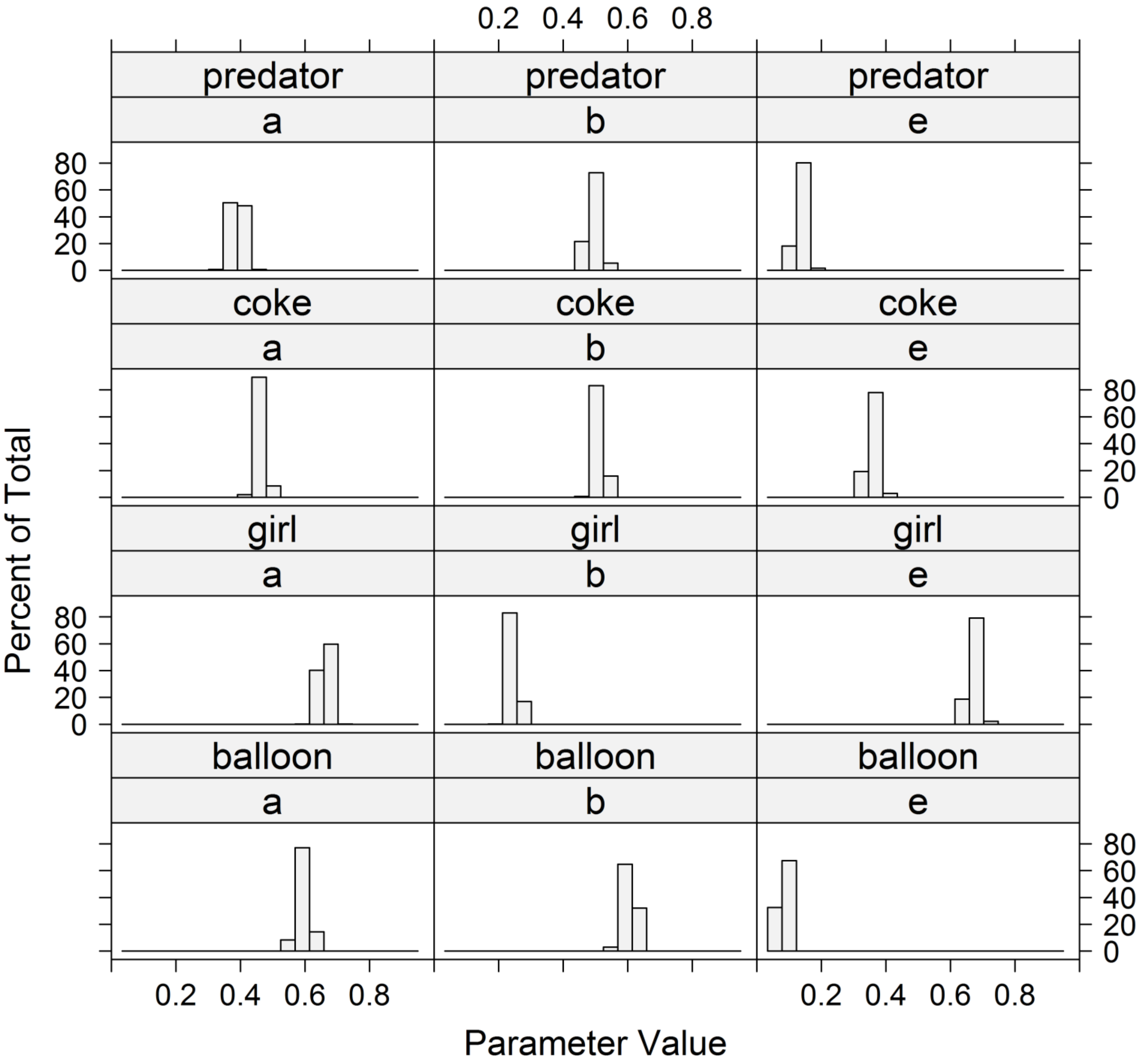
Endorsement
0.2
0.4
0.6
0.8
Inference

Simple model:



Black error bars: Range of individual level predictions from simple model



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EXPERIMENTAL PARADIGM

Reduced Inferences (Week 1)

A girl had sexual intercourse.

How likely is it that the girl is pregnant?

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Klauer, Beller, & Hütter (2010)
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Full Inferences (Week 2+)

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Full Inferences (Week 2)

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Role of conditional in Bayesian models:

- *PROB*: increases probability of conditional, $P(q|p)$ (Oaksford et al., 2000): $e' < e$
- *EX-PROB*: increases probability of conditional $P_{MP}(q|p) > P_{other}(q|p)$ (Oaksford & Chater, 2007)
- *KL*: increases $P(q|p)$ & Kullback-Leibler distance between $F_{p,q}$ and $F_{p,q}'$ is minimal (Hartmann & Rafiee Rad, 2012)

Consequence of updating: Effect is *content specific.*

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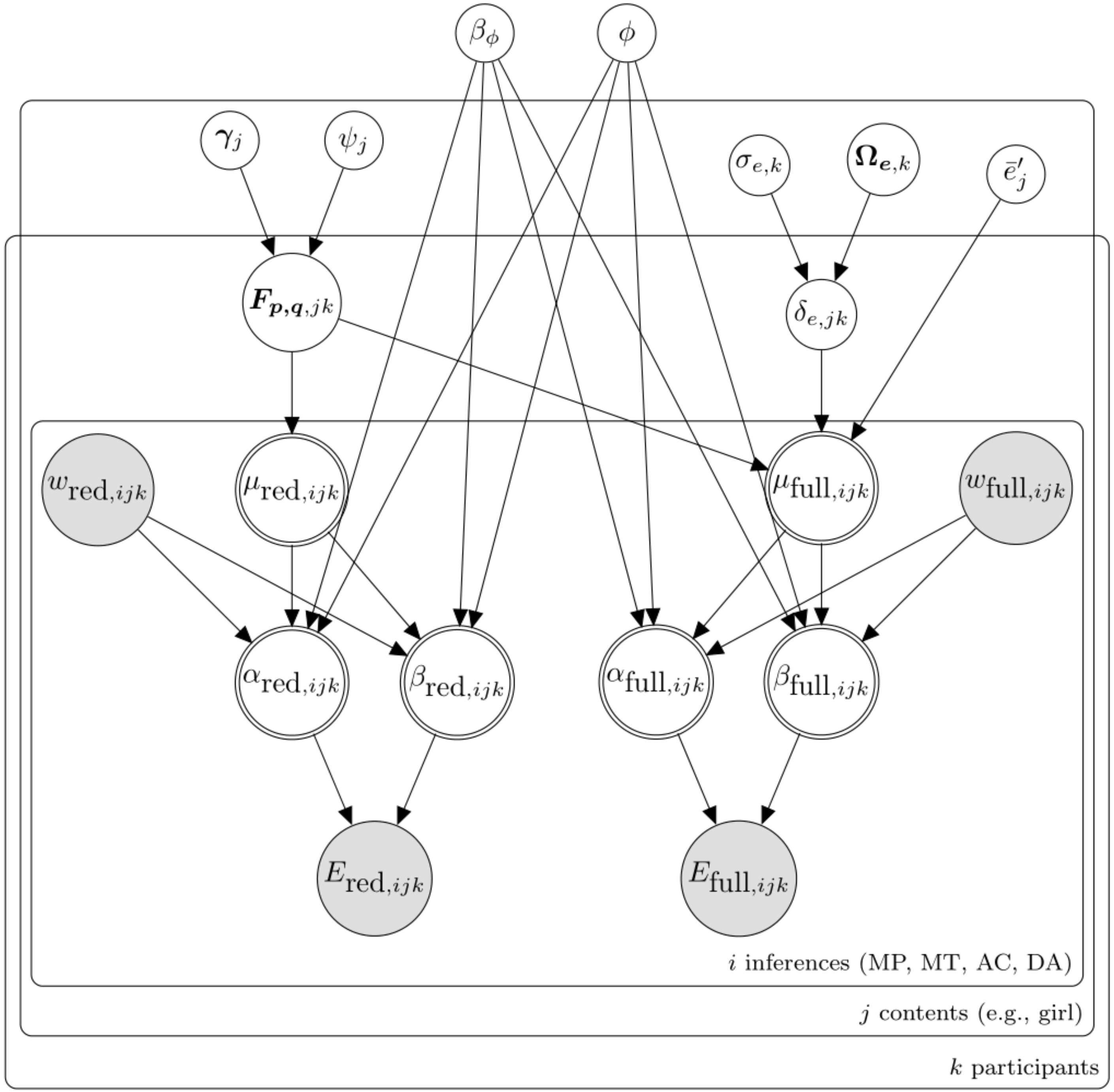
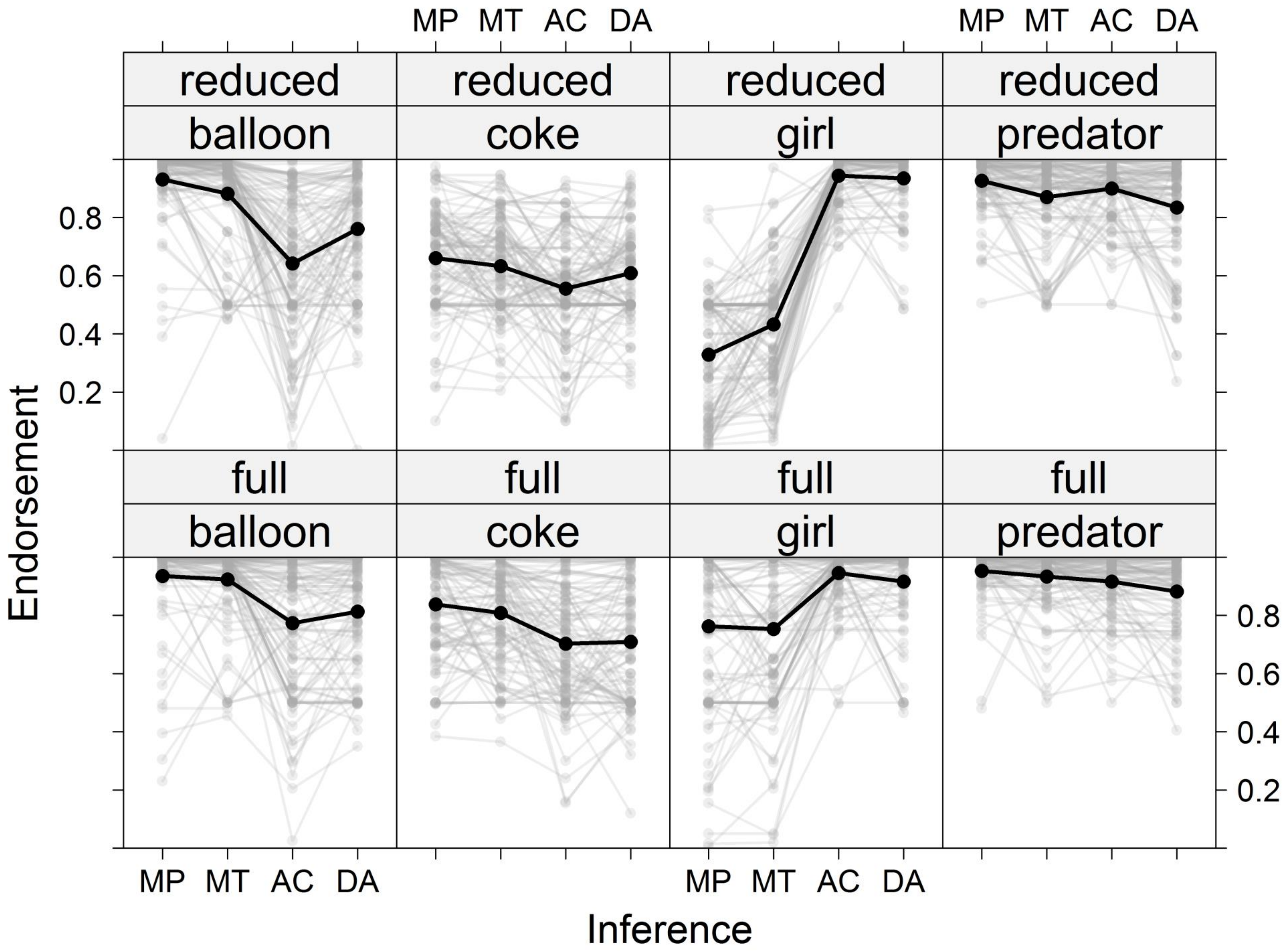
$$\boldsymbol{F}_{\boldsymbol{p},\boldsymbol{q},jk} \sim \text{Dirichlet}(\boldsymbol{\gamma}_j \times \psi_j)$$
$$\boldsymbol{\delta}_e \sim \text{MvNormal}(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{e}})$$

$$\boldsymbol{\Sigma}_e = \boldsymbol{\sigma}_e \boldsymbol{\Omega}_e$$
$$\boldsymbol{\Omega}_e \sim \text{LKJ}(1)$$
$$\sigma_{e,k} \sim \text{Cauchy}^{+}(0,4)$$
$$\bar{e}'_j \sim \text{Normal}(0,1)$$

$$l_{ij} = \max\left(\frac{a_{ij} - b_{ij}}{a_{ij}}, 0\right)$$
$$e'_{ij} = e_{ij} - \left((e_{ij} - l_{ij}) \times \Phi(\bar{e}'_j + \delta_{e,jk})\right)$$

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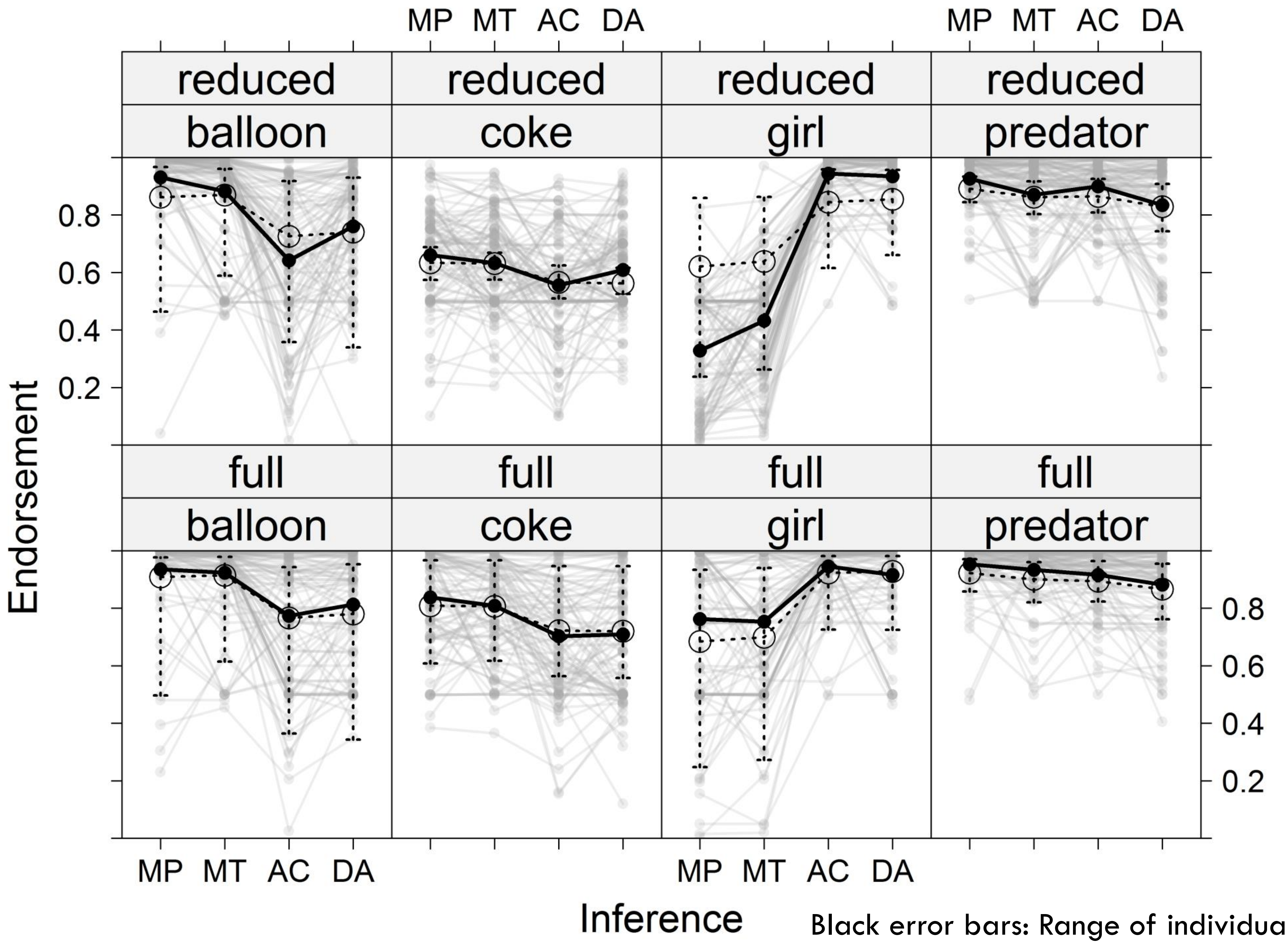

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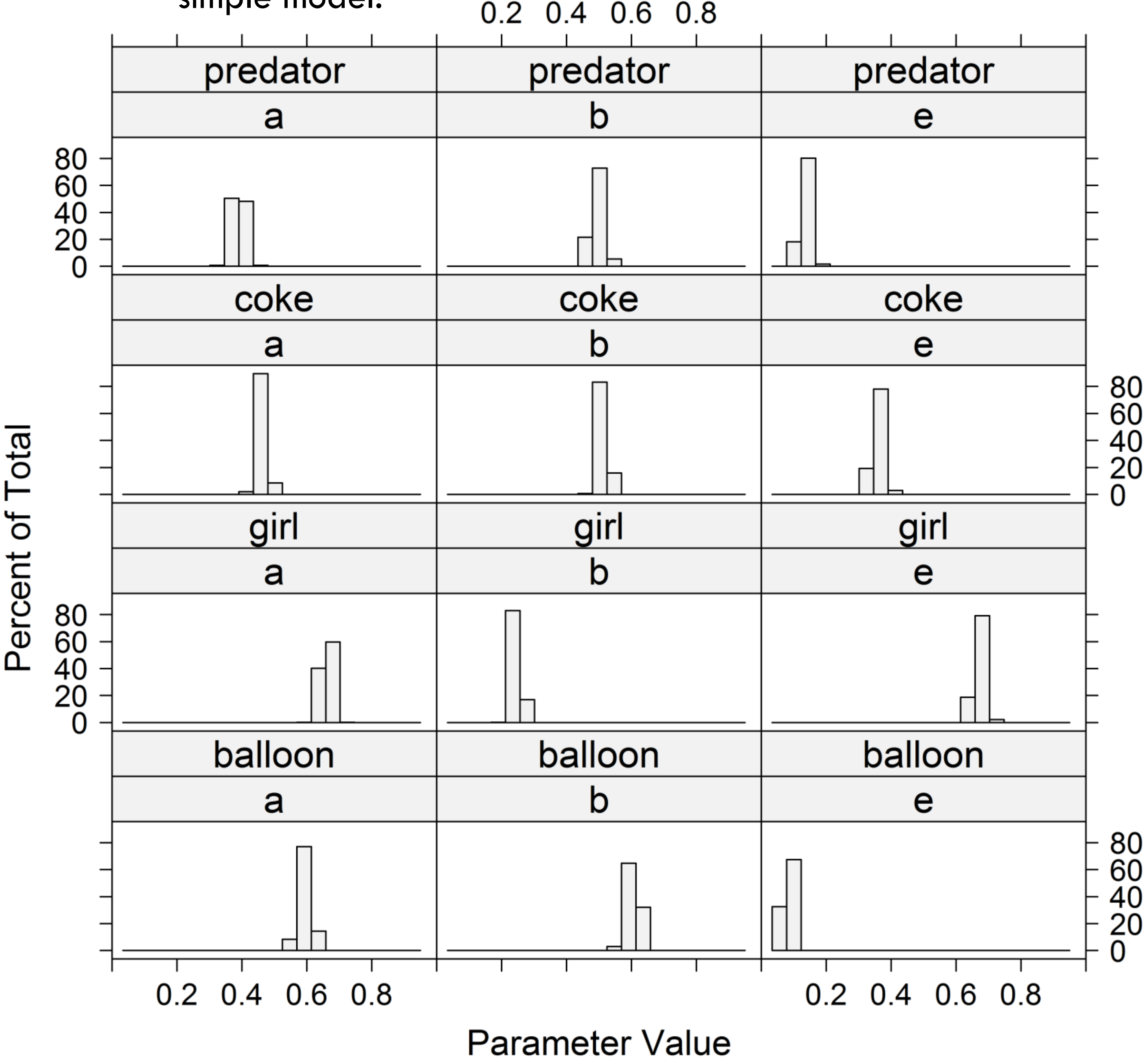

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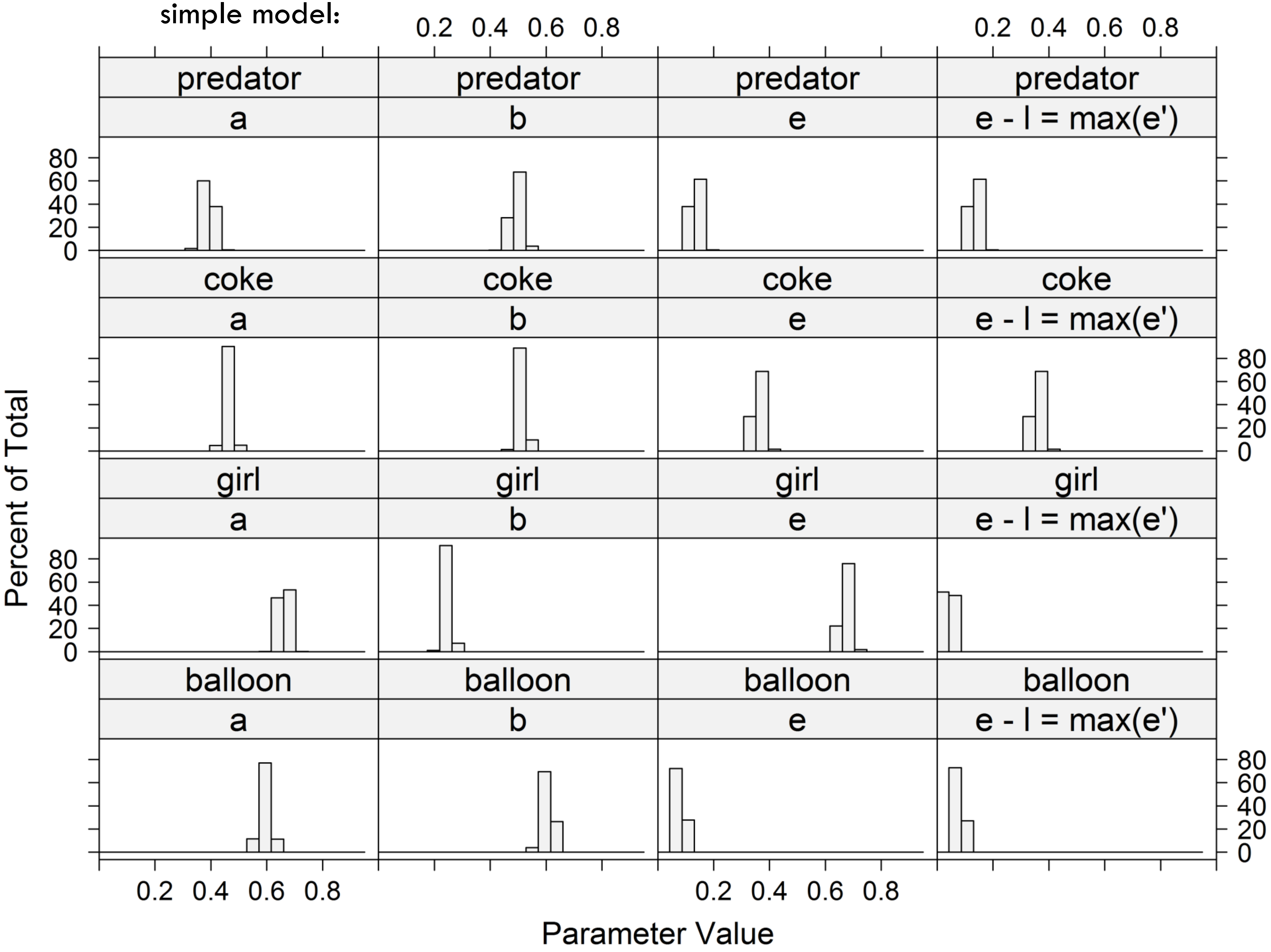

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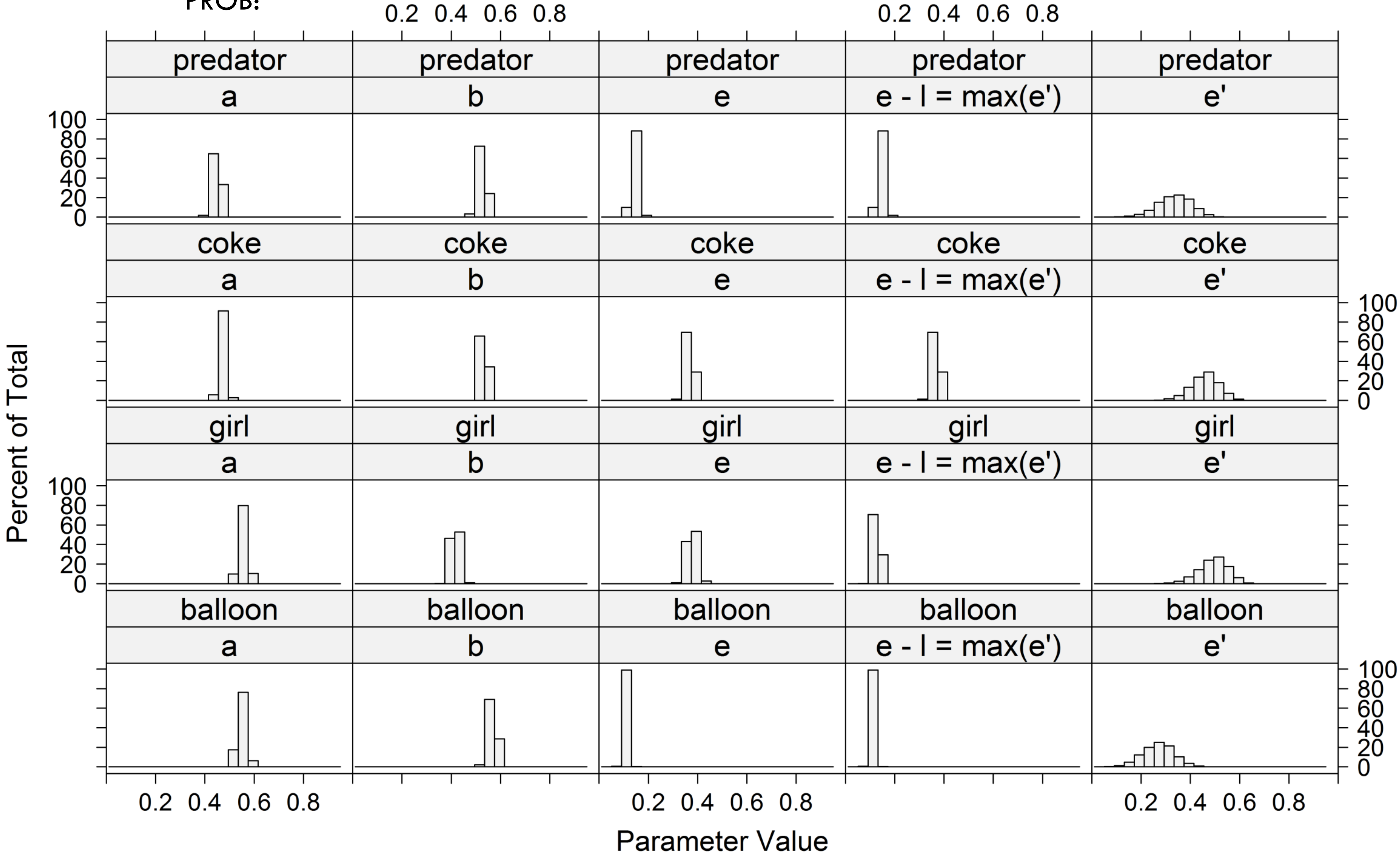
predator
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b
e
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e'
coke
girl
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If a girl had sexual intercourse, then she

Role of conditional in Bayesian models:

- ~~*PROB*: increases probability of conditional, $P(q|p)$ (Oaksford et al., 2000): $e' < e$~~
- ~~*EX-PROB*: increases probability of conditional $P_{MP}(q|p) > P_{other}(q|p)$ (Oaksford & Chater, 2007)~~
- *KL*: increases $P(q|p)$ & Kullback-Leibler distance between $F_{p,q}$ and $F_{p,q}'$ is minimal (Hartmann & Rafiee Rad, 2012)

Consequence of updating: Effect is *content specific*.

p	$P(p \wedge q)$	$P(p \wedge \neg q)$
$\neg p$	$P(\neg p \wedge q)$	$P(\neg p \wedge \neg q)$

p'	$P(p' \wedge q')$	$P(p' \wedge \neg q')$
$\neg p'$	$P(\neg p' \wedge q')$	$P(\neg p' \wedge \neg q')$

Kullback-Leibler (KL) Modell
Hartmann & Rafiee Rad (2012)
Singmann, Klauer, & Beller (2016, Exp. 1 & 3)

Parameterization of $\boldsymbol{F_{p,q}}$:
- $h = P(q)$
- $\alpha = P(q|p)$
- $\beta = P(q|\neg p)$

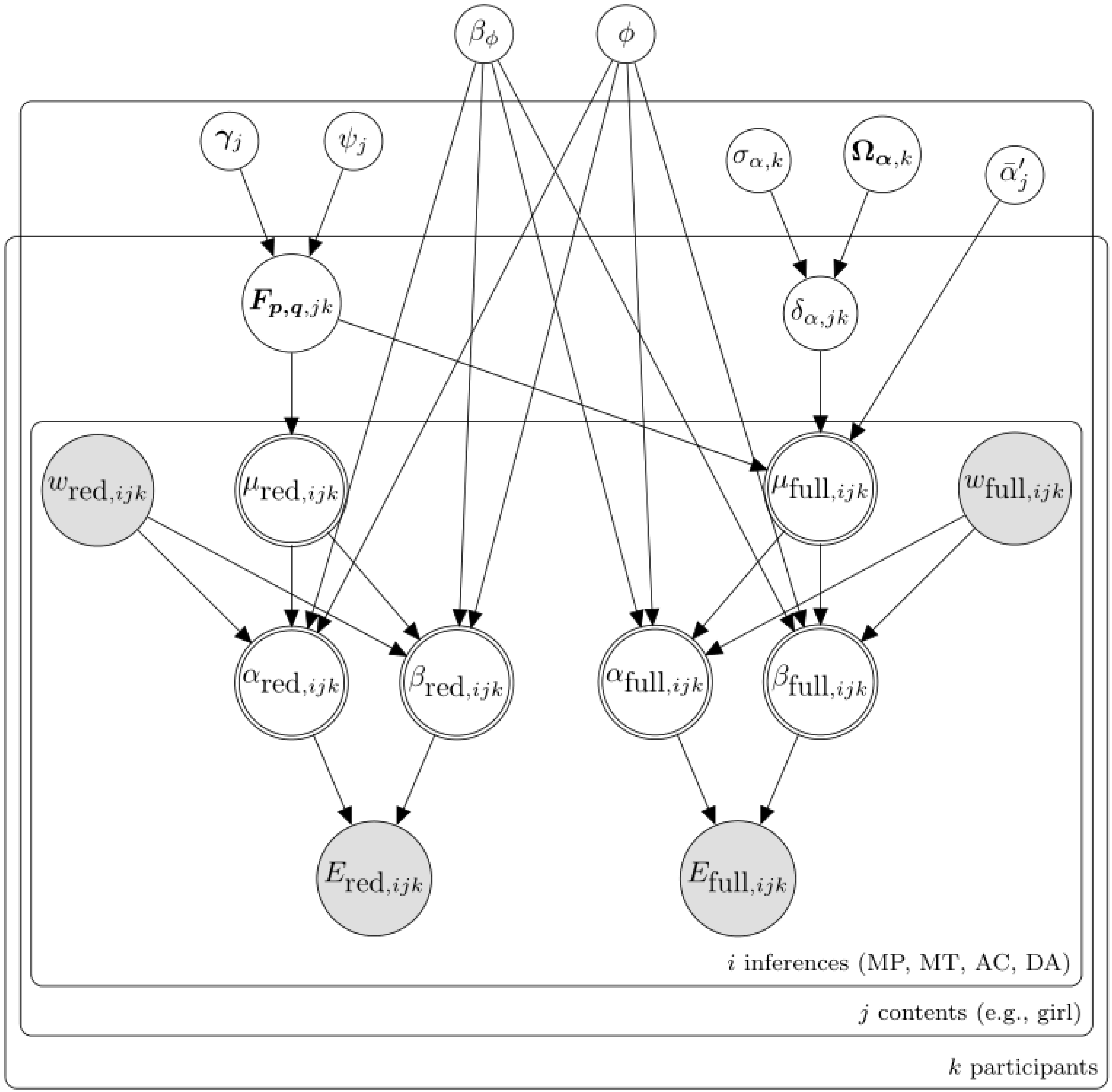
For $\boldsymbol{F_{p,q}}'$:
- $\alpha' > \alpha$
- Kullback-Leibler divergence between $\boldsymbol{F_{p,q}}$ and $\boldsymbol{F_{p,q}}'$ minimal.

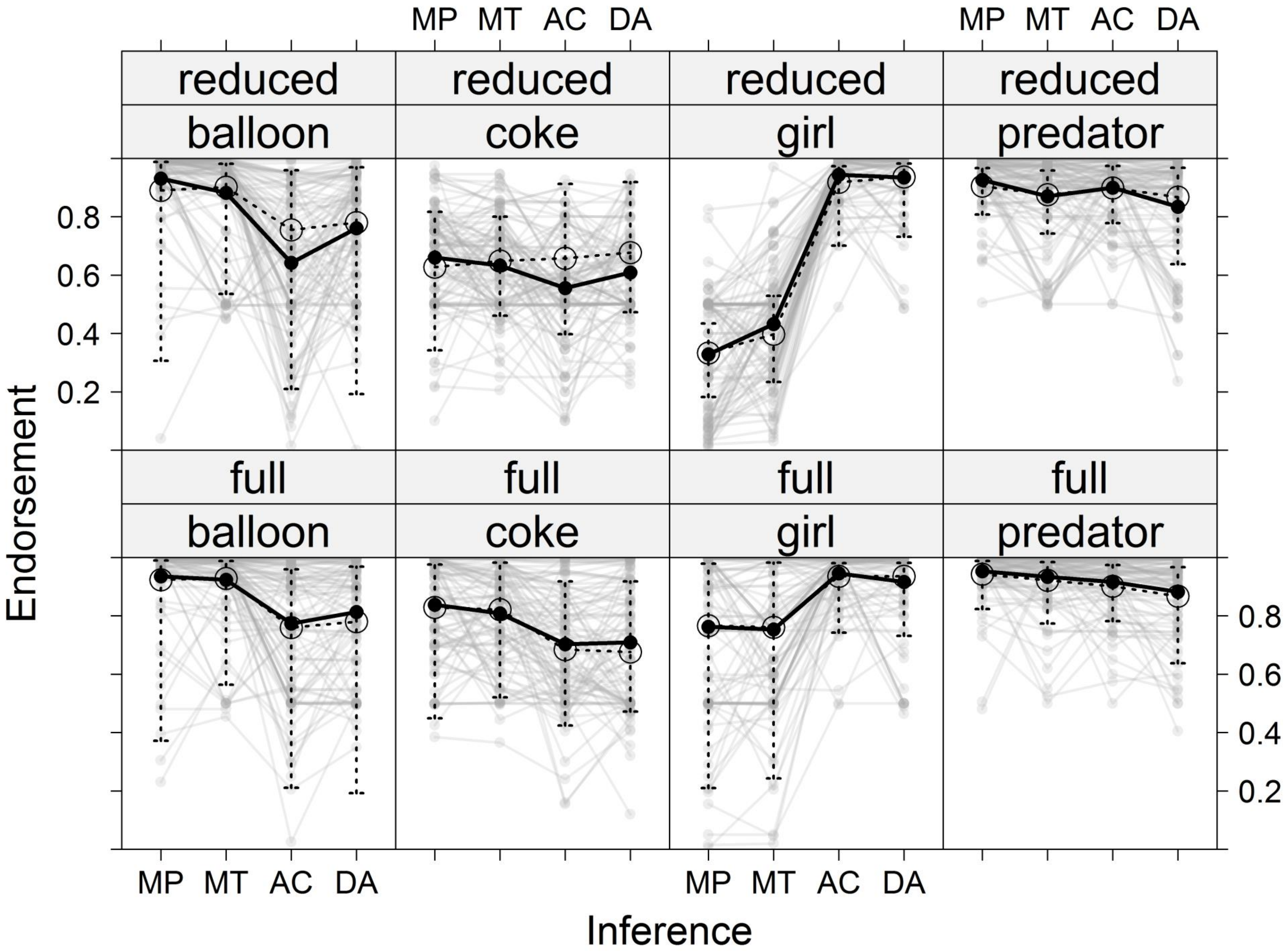
$$E_{\mathrm{red},ijk} \sim \mathrm{Beta}(\alpha_{\mathrm{red},ijk}, \beta_{\mathrm{red},ijk})$$
$$E_{\mathrm{full},ijk} \sim \mathrm{Beta}(\alpha_{\mathrm{full},ijk}, \beta_{\mathrm{full},ijk})$$

$$\boldsymbol{F_{p,q,jk}} \sim \mathrm{Dirichlet}(\boldsymbol{\gamma}_j \times \psi_j)$$
$$\boldsymbol{\delta}_\alpha \sim \mathrm{MvNormal}(\mathbf{0}, \boldsymbol{\Sigma_\alpha})$$

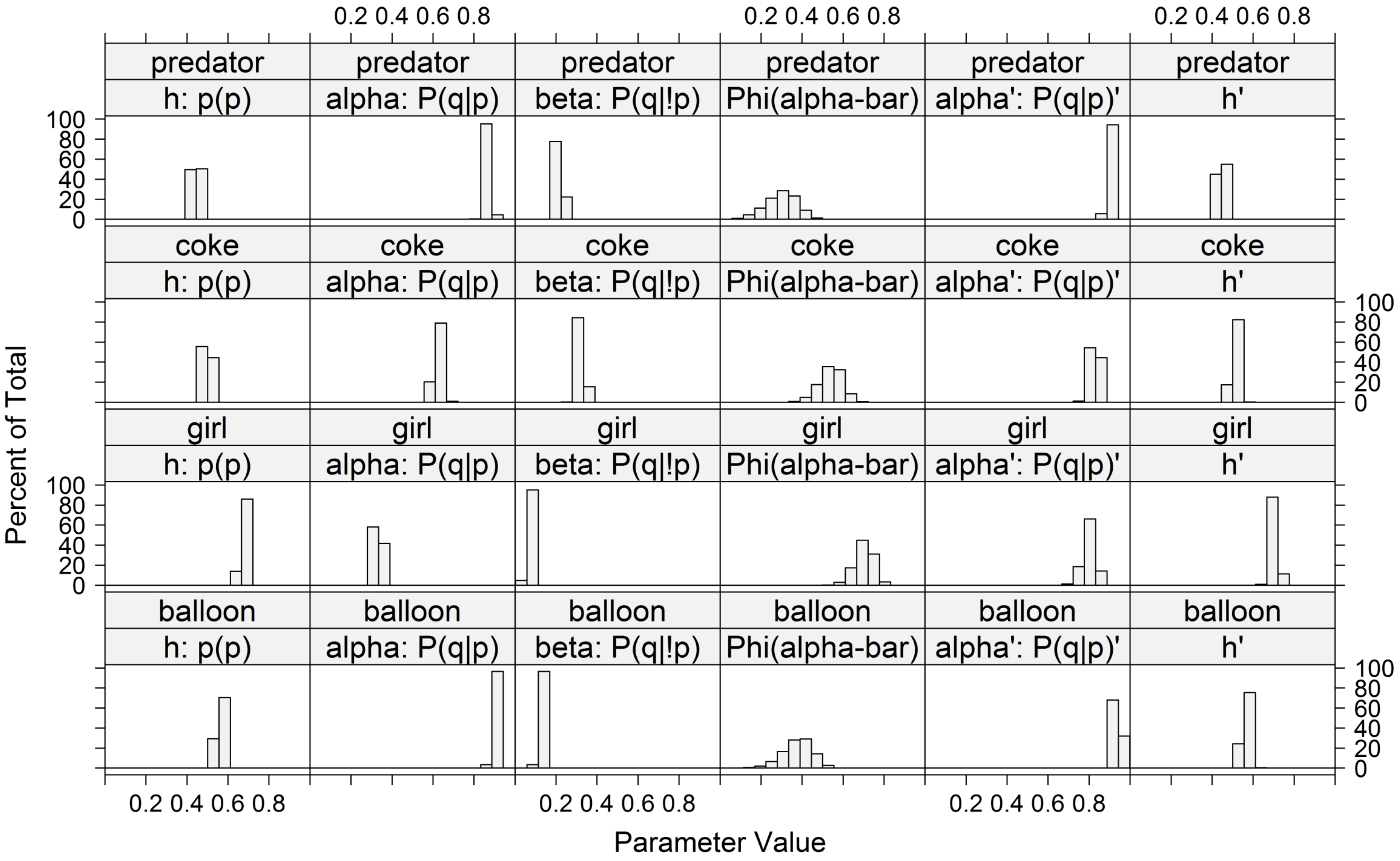
$$\boldsymbol{\Sigma}_\alpha = \boldsymbol{\sigma}_\alpha \boldsymbol{\Omega}_\alpha$$
$$\boldsymbol{\Omega}_\alpha \sim \mathrm{LKJ}(1)$$
$$\sigma_{\alpha,k} \sim \mathrm{Cauchy}^+(0,4)$$
$${}'_j \sim \mathrm{Normal}(0,1)$$

$$\alpha'_{ij} = \alpha_{ij} + (1 - \alpha_{ij}) \times \Phi(\bar{\alpha}'_j + \delta_{\alpha,jk}))$$





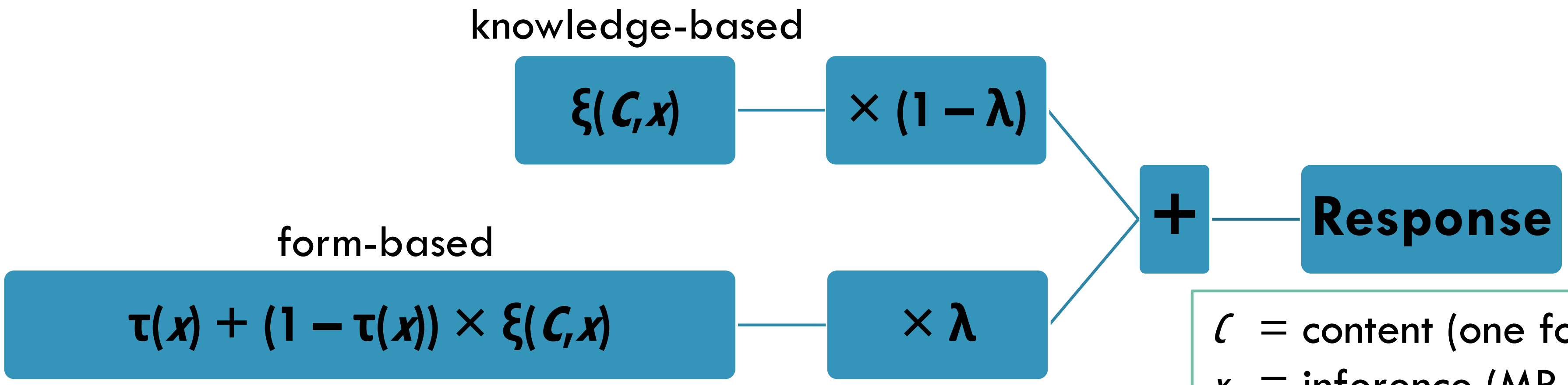
MP MT AC DA
reduced
balloon
reduced
coke
reduced
girl
reduced
predator
full
balloon
full
coke
full
girl
full
predator
0.8
0.6
0.4
0.2
Endorsement
Inference


predator
h: p(p)
predator
alpha: P(q|p)
predator
beta: P(q|!p)
predator
Phi(alpha-bar)
predator
alpha': P(q|p)'
predator
h'
coke
h: p(p)
coke
alpha: P(q|p)
coke
beta: P(q|!p)
coke
Phi(alpha-bar)
coke
alpha': P(q|p)'
coke
h'
girl
h: p(p)
girl
alpha: P(q|p)
girl
beta: P(q|!p)
girl
Phi(alpha-bar)
girl
alpha': P(q|p)'
girl
h'
balloon
h: p(p)
balloon
alpha: P(q|p)
balloon
beta: P(q|!p)
balloon
Phi(alpha-bar)
balloon
alpha': P(q|p)'
balloon
h'
0.2 0.4 0.6 0.8
0 20 40 60 80 100
Percent of Total
Parameter Value

DUAL-SOURCE MODEL (DSM)

Klauer, Beller, & Hütter (2010, Exp. 1)
Singmann, Klauer, & Beller (2016, Exp. 1 & 3)

Par.	Interpretation	Influencing Factors
λ	Relative weight given to form-based versus knowledge-based evidence	E.g., speaker expertise, instructional emphasis on rule
τ	Degree to which an inference is seen as logically warranted	E.g., inference (MP, MT, AC, DA), connective (e.g., "if -then" vs. "or")
ξ	Knowledge-based response proposal	E.g., contents of the premises/ salience of counterexamples

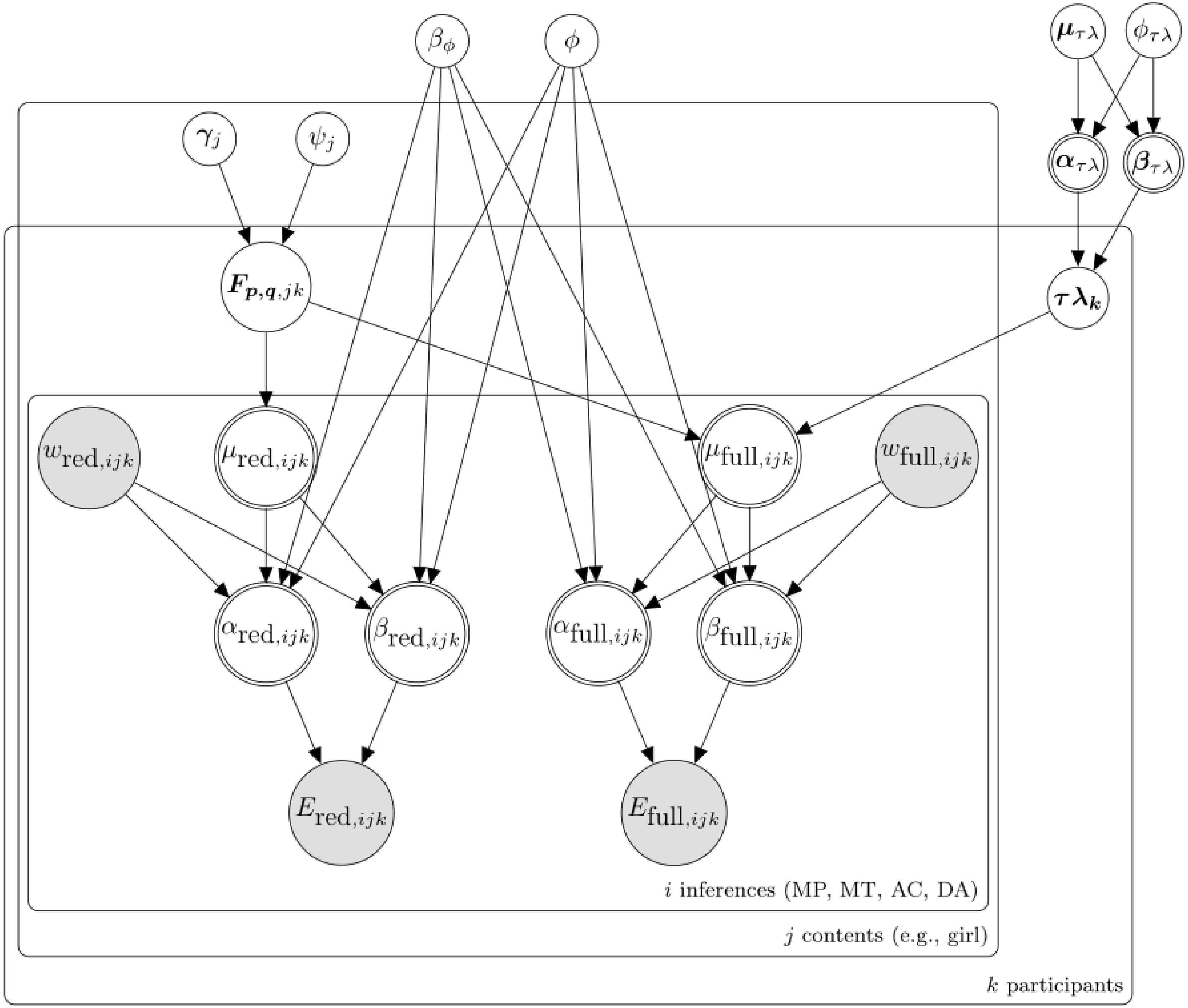


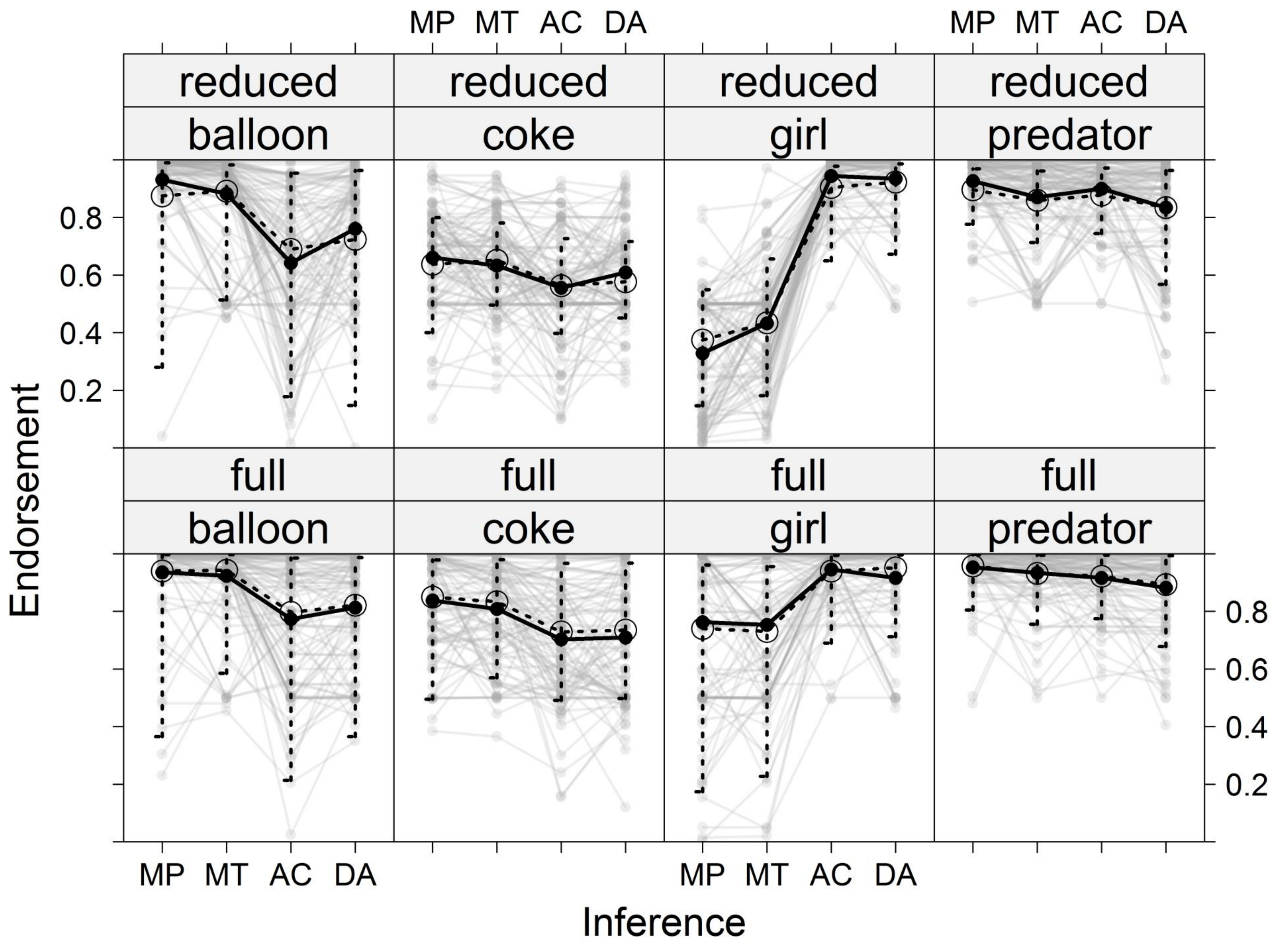
C = content (one for each p and q)
x = inference (MP, MT, AC, & DA)

$$E_{\text{red},ijk} \sim \text{Beta}(\alpha_{\text{red},ijk}, \beta_{\text{red},ijk})$$
$$E_{\text{full},ijk} \sim \text{Beta}(\alpha_{\text{full},ijk}, \beta_{\text{full},ijk})$$

$$\boldsymbol{\tau\lambda_k} \sim \text{Beta}(\boldsymbol{\alpha}_{\tau\lambda}, \boldsymbol{\beta}_{\tau\lambda})$$
$$\boldsymbol{\mu}_{\tau\lambda} \sim \text{Uniform}(0, 1)$$
$$\phi_{\tau\lambda} \sim \text{Cauchy}^{+}(0, 5)$$

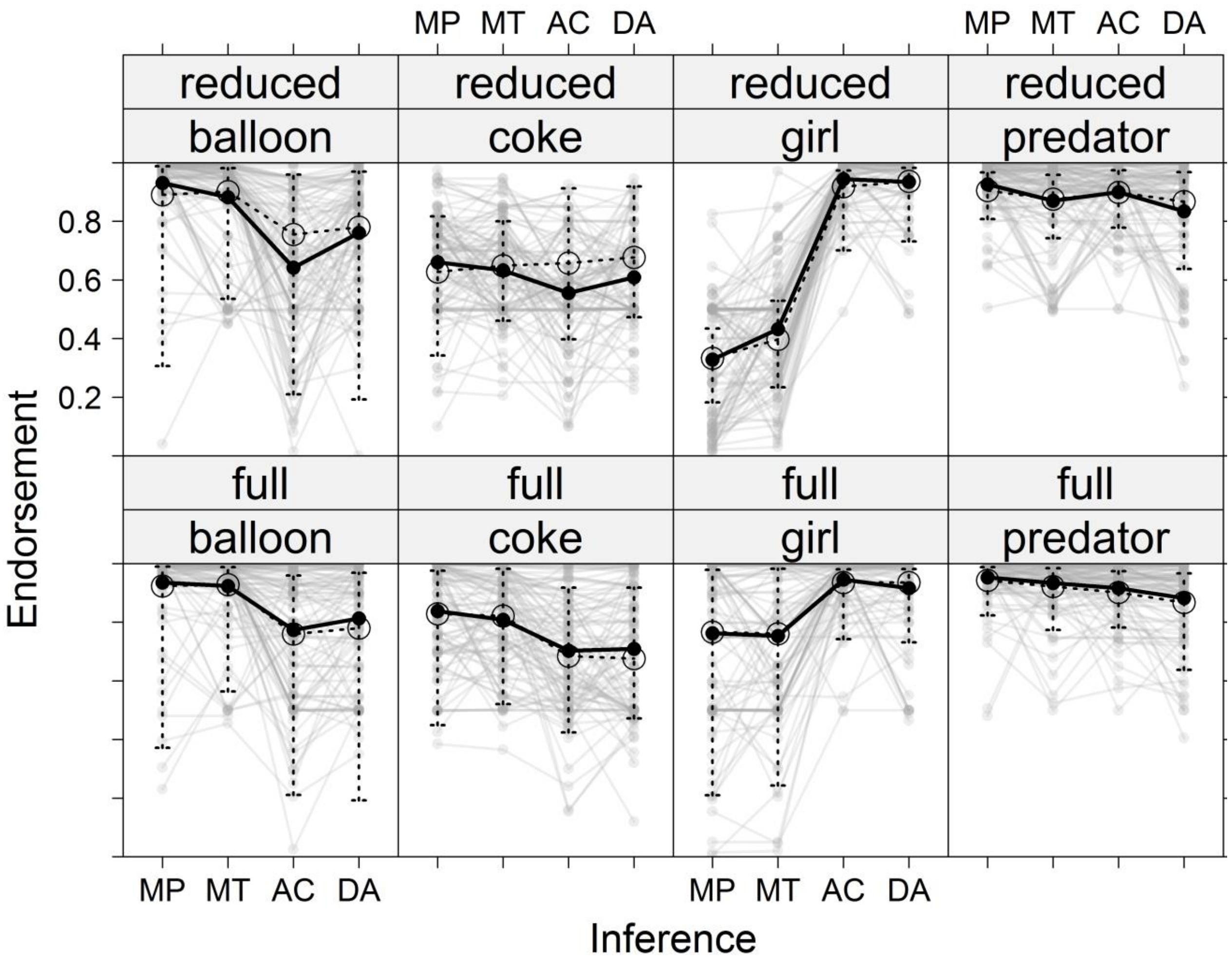
$$\lambda_k = \max(\boldsymbol{\tau\lambda}_k)$$
$$\boldsymbol{\tau}_k = \frac{\boldsymbol{\tau\lambda}_k}{\lambda_k}$$





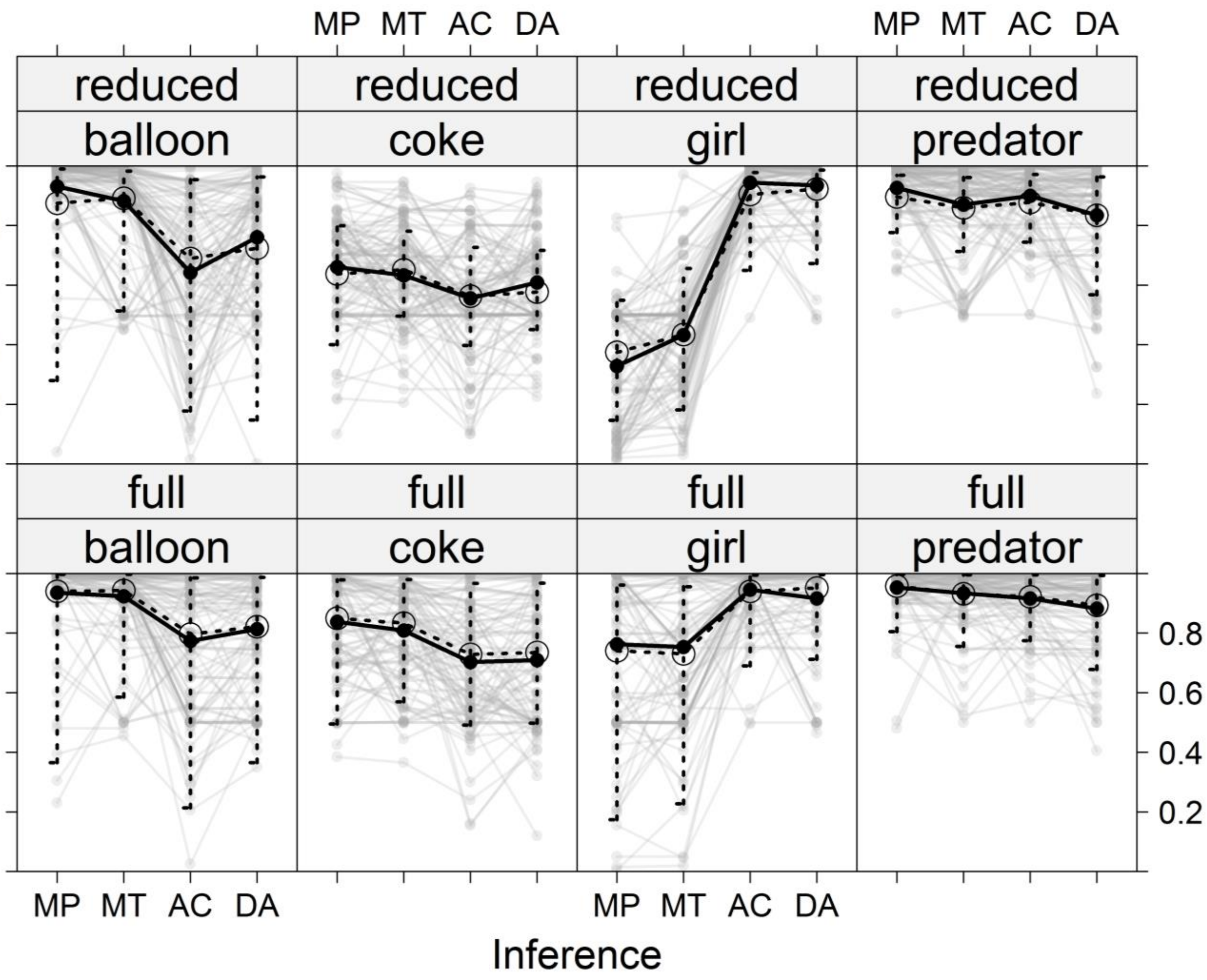
reduced
balloon
reduced
coke
reduced
girl
reduced
predator
full
balloon
full
coke
full
girl
full
predator
MP MT AC DA
Endorsement
0.2 0.4 0.6 0.8
Inference

KL Model:



reduced balloon
reduced coke
reduced girl
reduced predator
full balloon
full coke
full girl
full predator
MP MT AC DA
0.2
0.4
0.6
0.8
Endorsement
Inference

DSM:



reduced balloon
reduced coke
reduced girl
reduced predator
full balloon
full coke
full girl
full predator
MP MT AC DA
0.2
0.4
0.6
0.8
Inference

SUMMARY: HIERARCHICAL BAYESIAN IMPLEMENTATION OF BAYESIAN MODELS OF REASONING

Bayesian statistics offer:

- Principled approach to model individual differences
- Allows investigation of individual level and group-level parameters
- Provides additional information (e.g., precision of probability distribution estimates, correltaion among individual parameters)

For inferences without conditional (i.e., purely knowledge) a simple Bayesian model provides good account.

Learning a conditional can be modeled with:

- Bayesian model that assumes unconsrained updating of P(q|p) and KL minimization (Hartmann & Rafiee Rad, 2012).
- Dual-Source Model (Klauer et al., 2010; Singmann et al., 2016), which assumes individuals combine background knowledge with the subjective probability with which they see a specific inference as logically warranted.

THAT WAS ALL

$F_{p,q}$: If a balloon is pricked with a needle then it will pop.		
ψ_j: 15 [27]	q	¬q
p	.36	.06
¬p	.16	.42

$F_{p,q}$: If a person drinks a lot of coke then the person will gain weight.		
ψ_j: 58 [250]	q	¬q
p	.29	.17
¬p	.23	.31

$F_{p,q}$: If a girl has sexual intercourse then she will be pregnant.		
ψ_j: 33 [21]	q	¬q
p	.24 [.35]	.41 [.21]
¬p	.03 [.07]	.31 [.37]

$F_{p,q}$: If a predator is hungry then it will search for prey.		
ψ_j: 46 [130]	q	¬q
p	.51	.06
¬p	.07	.36

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p	.29	.17
¬p	.23	.31

Precision of group-level parameter for $F_{p,q}$ (ψ_j), initial model:

10.5 [8.2, 13.2]

27.1 [20.3, 37.0]

19.3 [13.8, 27.3]

27.3 [20.1, 36.9]

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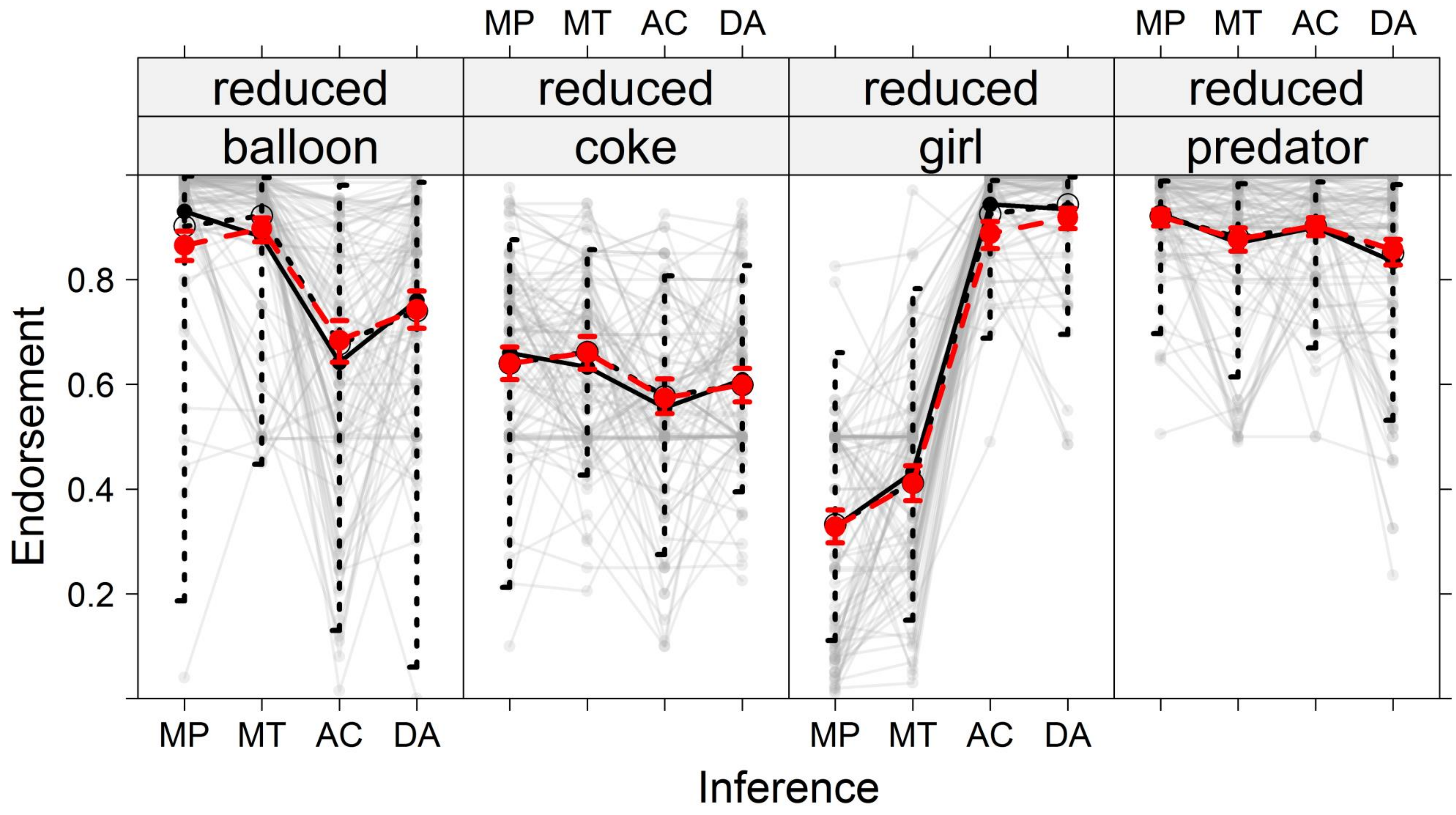
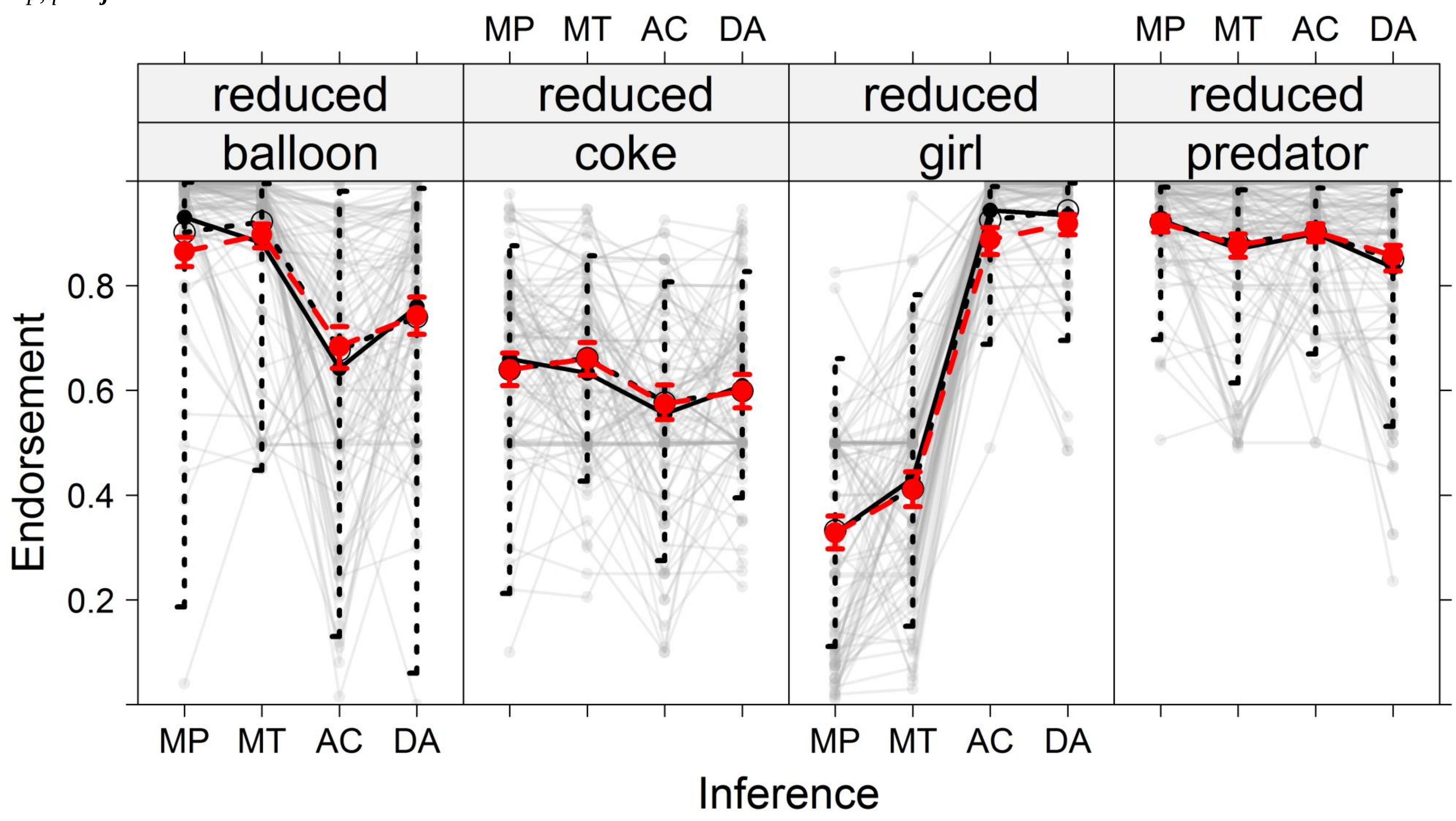


Black error bars: Range of individual level predictions from simple model

Precision of group-level parameter for $F_{p,q}$ (ψ_j): 10.5 [8.2, 13.2] 27.1 [20.3, 37.0] 19.3 [13.8, 27.3] 27.3 [20.1, 36.9]



Black error bars: Range of individual level predictions from simple model